Clustering of noise-induced oscillations

O. V. Sosnovtseva, A. I. Fomin, D. E. Postnov, and V. S. Anishchenko
Physics Department, Saratov State University, Astrakhanskaya Strasse 83, Saratov, 410026, Russia
(Received 23 January 2001; published 17 July 2001)

The subject of our study is clustering in a population of excitable systems driven by Gaussian white noise and with randomly distributed coupling strength. The cluster state is frequency-locked state in which all functional units run at the same noise-induced frequency. Cooperative dynamics of this regime is described in terms of effective synchronization and noise-induced coherence.

DOI: 10.1103/PhysRevE.64.026204 PACS number(s): 05.45.Ac, 05.40.−a

When self-sustained systems are coupled their cooperative behavior reveals a set of dynamical patterns of which the most interesting ones are clustering, coherent structures, and synchronization. Synchronization effects have also been observed in stochastic systems where noise controls the characteristic frequency of the system [1–3] and may enhance synchronization [4,5]. Array-enhanced coherence behavior has been also intensively studied in noisy extended systems [6–8].

Clustering, i.e., formation of group of functional units with similar properties (amplitudes, phases or frequencies) is an important phenomenon that is assumed to underly perception and processing of the information by the brain [9], for instance. The problem of clustering was formulated and analyzed in general context in the framework of the phase equations [10], of self-sustained periodic oscillators [11], of chaotic dynamical networks [12], and of a chain of bistable elements [13]. Vadivasova *et al.* [14] showed that cluster synchronization is structurally stable to small fluctuations.

In this paper we investigate how clustering occurs in a population of nonhomogeneous excitable systems with randomly distributed coupling strength. These excitable systems are specially sensitive to external noise demonstrating coherence resonance [15,16] at appropriate amount of random forcing. Bearing this in mind, we consider spatial collective response of such functional units in terms of effective synchronization and as regularization of noise-induced oscillations with distinct eigenfrequencies.

Let us take Fitz Hugh-Nagumo model as the unit in an array. Being originally suggested for the description of nerve pulses [17], it models excitable dynamics in different fields ranging from chemical reactions to biological processes [18].

With x and y being a fast and a slow variable, respectively, this dynamics reads

$$\epsilon \frac{dx_j}{dt} = x_j - \frac{x_j^3}{3} - y_j + g_j(x_{j+1} + x_{j-1} - 2x_j), \qquad (1)$$

$$\frac{dy_j}{dt} = x_j + a_j + D\xi_j(t), \quad j = 1, \ldots, N.$$

Here, $\epsilon = 0.01$ is the small time scale ratio of the two variables, the parameter a governs the character of solutions and is responsible for excitory properties of the individual dynamics. We use free boundary and random initial conditions.

Collective dynamics of assembly of coupled oscillators achieves high importance in biomedical applications. Typically, a population of identical units with the same coupling properties serves as the simplest model under consideration. In nature, however, full identity in properties and operating conditions would be idealization. In our work, in the contrast to the previous studies, we investigate ordering effects in assemblies of elements that are (i) nonhomogeneous, i.e., the activation parameters a_j are random numbers distributed uniformly on [1.0;1.1]; (ii) subjected to stochastic forcing by Gaussian white noise $\xi_j(t)$ that is statistically independent in space and with zero mean value, i.e., $\langle \xi_j(t)\xi_i(t') \rangle = \delta_{ij}\delta(t - t')$ and $\langle \xi_j(t) \rangle = 0$; (iii) coupled with the strengths g_j that has random uniform distribution on some range of Δ width [19].

Thus, our model provides disorder between interacting units in different ways. Interesting question is how such elements adjust their motions in accordance to one another to reach some kind of coherence?

In the case of a single excitable system driven by moderate amount of noise, the trajectory can become quite regular, a phenomenon known as coherence resonance (CR) [16]. The Fourier power spectrum possesses a well-defined global maximum. The quantitative measure of coherence resonance is so-called regularity that can be calculated as [16] $R_j = \langle \tau_j \rangle / \sqrt{Var(\tau_j)}$, where the pulse duration τ_j is specified as the sum of activation time needed to excite the system from the stable fixed point and excursion time needed to return from the excited state to the fixed point. Time averaged pulse duration identifies the mean period and, hence, the mean frequency $\langle f_j \rangle = 1/\langle \tau_j \rangle$ of noise-induced oscillations. Thus, with coherence resonance effect, a noise-driven excitable system can be considered as a some kind of stochastic oscillator whose behavior can be described in terms of a noise-induced eigenfrequency [3] and a phase introduced as the position on a stochastic limit cycle [20].

In our experiments with varying distribution interval of coupling strength and for a certain level of noise, three basic types of space-time behavior in one-dimensional array (1) of 100 units can be observed. For a vanishing and very narrow Δ the behavior is totally incoherent that is reflected in irregular pattern of black (firing state) and white colors [Fig. 1(a)]. The firing events in individual units occur at different eigenfrequences which are randomly spreaded in the range [0.05;0.27] [Fig. 1(b)]. In this case, no stable frequency- or phase-locked groups can be detected. The qualitatively different behavioral pattern we encounter for broader range of coupling [Figs. 1(c) and 1(d)]. The synchronized groups, i.e., clusters of stochastic elements, appear. Within each cluster

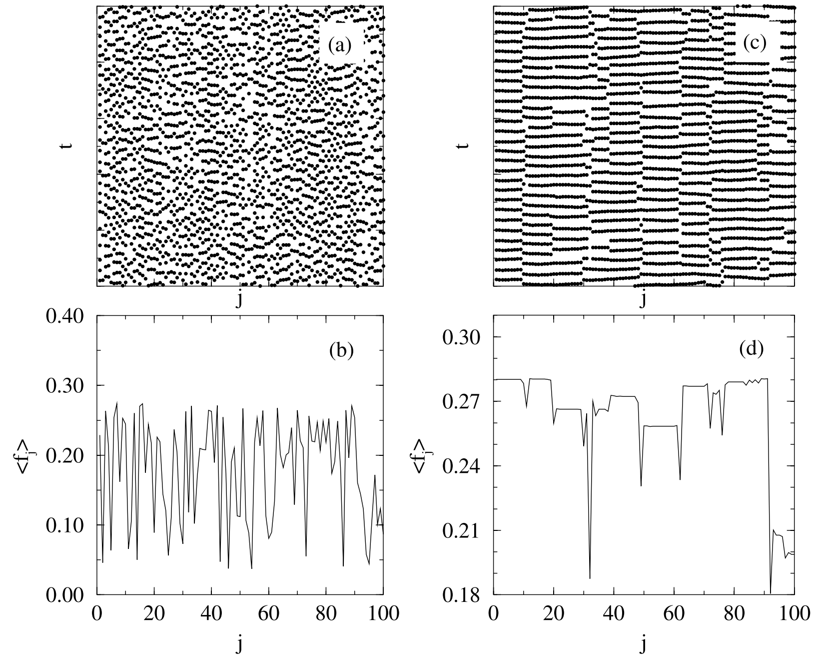

FIG. 1. Spatiotemporal evolution and eigenfrequencies $\langle f_j \rangle$ of an array of 100 excitable units at $D=0.025$ for different width of coupling range $\Delta=0.002$ (a,b) and for $\Delta=0.1$ (c,d). A sequence of clusters is clearly seen in the latter case. Black dots indicate firing events.

the frequency difference between any two oscillators vanishes or is small in comparison with the difference between neighboring clusters. With increasing distribution interval of coupling, the number of clusters is decreased (Fig. 2) while finally the global synchronous state (one-cluster state) where all units fire simultaneously, is achieved. Since the incoherent behavior and the totally synchronized behavior are well understood [5,8], we focus our study on clustering of noise-induced oscillations.

Let us consider now an individual cluster as a spatial meta unit of an array and describe its main properties. Because of the given distribution of system parameters, the elements in cluster have different randomly scattered frequencies for vanishing coupling, i.e., there is no correlation between firing events of different cells. With interaction, a frequency locking effect that is responsible for cluster formation takes place [Fig. 3(a)]. In this case, elements composed cluster display regular synchronous firings.

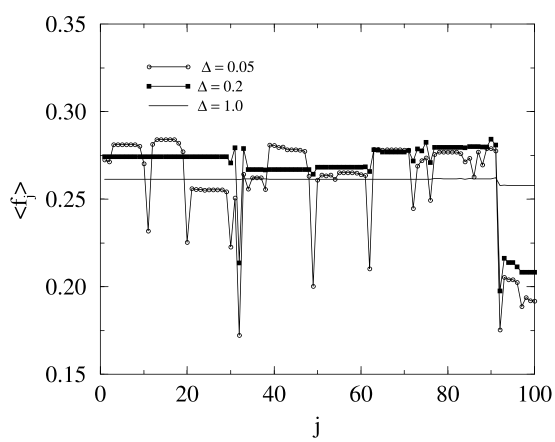

FIG. 2. Reduction of the number of frequency-locked clusters with increasing width of coupling range $(D=0.025)$.

However, the deviation of pulse duration $\sigma_j^2=\langle \tau_j^2 \rangle - \langle \tau_j \rangle^2$ is changed within a cluster. It is minimum in the center of cluster and the difference in σ_j between neighboring elements increases at the ends of cluster [Fig. 3(b)]. Thus, with frequency entrainment, oscillators demonstrate different degree of mutual synchronization.

Frequency-locking entrainment is closely related to the phase conditions. For stochastic systems one has to use the notion of ''effective synchronization'' [21]. In the presence of Gaussian noise (or another random process with unlimited distribution function) the phase-locked state has to be inevitably broken at some moment. Thus, the system is supposed to be effectively synchronized if the phase locking is ob-

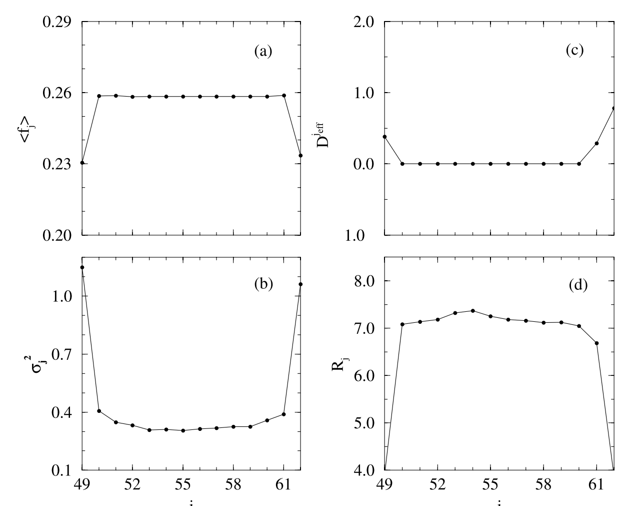

FIG. 3. Eigenfrequency $\langle f_j \rangle$ (a), deviation of pulse duration σ_j^2 (b), effective cross-diffusion coefficient D_{eff}^j (c), and the noise-induced regularity R_j (d) within a single cluster. Widths of coupling interval and noise intensity are fixed at 0.1 and 0.025, respectively.

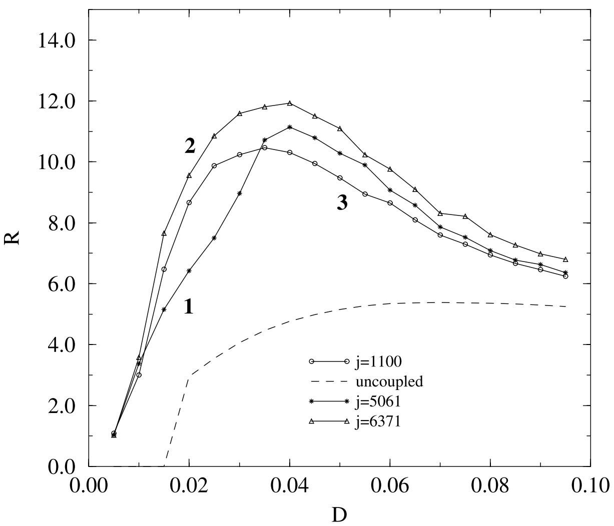

FIG. 4. Illustration of synchronization-enhanced coherence resonance for the system (1) demonstrating cluster structure for $\Delta = 0.1$. The regularity averaged over spatial coordinate R is plotted versus noise intensity for individual clusters (curves 1 and 2) and for the whole array with a cluster structure (curve 3). The dashed curve corresponds to the uncoupled array.

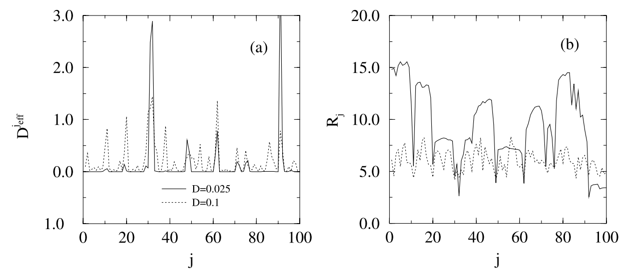

FIG. 5. Synchronous (a) and coherence (b) properties along the array with cluster structure for varying level of noise. The width of coupling interval is fixed at 0.1.

served during a finite but long enough time determined *a priori*. A measure of stochastic synchronization is the cross-diffusion coefficient $D^j_{eff} = \frac{1}{2} d/dt[\langle \phi_j^2(t)\rangle - \langle \phi_j(t)\rangle^2]$ [22]. This quantity describes the spreading in time of an initial distribution of the phase difference $\phi_j(t)$ [23] between the neighboring elements. In our study, cross-diffusion coefficient keeps vanishing value within each cluster [Fig. 3(c)] and gets different nonzero values for intercluster units. It fits the stronger condition of phase synchronization that provides high degree of collective entrainment within cluster of stochastic oscillators. Hence, the notion of effective synchronization can be generalized to the spatially extended group of elements. Similar effect has been observed for coupled Van der Pol oscillators with fluctuations [14].

What are the coherence properties of such frequency-locked clusters? It is clearly seen that the regularity exhibits maximum value within synchronized state [Fig. 3(d)], while the outer-cluster elements demonstrate the lower level of coherence. Comparative analysis of the regularity and pulse deviation functions allows us to assume that high coherence behavior within a cluster is related to synchronization phenomenon.

In general, the collective response of a cluster is characterized by two aspects. The first one is synchronization effect that leads to the frequency and phase entrainment. The second one is the regularity of each functional unit due to CR effect. Remarkably, the regularity averaged over spatial coordinate can be maximized within each cluster by tuning the noise (Fig. 4). At weak external noise, a cluster considered as a whole functional unit demonstrates weak coherence in spite of firings in the elements of cluster can occur simultaneously. This is related to the relatively large fluctuations of the pulse duration of each composed elements. With the increasing D the coherence of temporal and spatial structure of firing process is enhanced and reaches a maximum. At large noise, the frequency and phase fluctuations grow rapidly and it leads to the destruction of coherence properties of composed units and, hence, of the spatial coherence structure. Because of the phenomenon of array-enhanced coherence resonance [8], the regularity of the whole cluster is much higher than that of uncoupled elements (compare the curves 1, 2, and the dashed curve).

Let us turn back to the whole system. Now an array composed by excitable elements can be considered in macro level as a sequence of clusters whose size and structure is determined by random distribution of firing properties and the degree of interaction. Figure 5 illustrates the ordering effect caused by the stochastic synchronization and the resulting high coherence within each cluster at the optimal level of noise. The coherence of net output is averaged over a set of clusters. Because of frequency difference between clusters, the regularity of array output is lower than the maximum value of each cluster (curve 3 in Fig. 4).

In summary, we first investigate coherence properties in an assembly of diffusively coupled excitable systems in a regime of cluster stochastic synchronization. Random distribution of system parameters responsible for excitory properties and strength of interaction leads to self-organization in the form of clustering that manifests itself as stochastic phase locking and as the mean frequency entrainment between a group of cells. Composed by a number of elements with different properties, each cluster can be considered as a "spatial" excitable unit exhibiting coherence resonance. Its degree of coherence can be enhanced by tuning the noise intensity. Gain of regularity within each cluster is associated with the effect of stochastic synchronization. We believe that these effects can be of importance for biological applications where the background noise may play a constructive role in ordering phenomena in a large networks of excitable elements through the synchronization mechanisms.

We thank T.E. Vadivasova for fruitful discussions. This paper is based upon work supported by the Naval Research Laboratory under Contract No. N68171-00-M-5430. The authors acknowledge Award No. REC-006 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF). D.E.P. acknowledges Grant No. RFBR N 01-02-16709.

[1] A.B. Neiman, Phys. Rev. E **49**, 3484 (1994).
[2] B.V. Shulgin, A.B. Neiman, and V.S. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1995).
[3] S.K. Han, T.G. Yim, D.E. Postnov, and O.V. Sosnovtseva, Phys. Rev. Lett. **83**, 1771 (1999).
[4] A. Neiman, A. Silchenko, V. Anishchenko, and L. Schimansky-Geier, Phys. Rev. E **58**, 7118 (1998).
[5] A. Neiman, L. Schimansky-Geier, A. Cornell-Bell, and F. Moss, Phys. Rev. Lett. **83**, 4896 (1999).
[6] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, and A.R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995).
[7] D.E. Postnov, S.K. Han, T.G. Yim, and O.V. Sosnovtseva, Phys. Rev. E **59**, R3791 (1999).
[8] B. Hu and C. Zhou, Phys. Rev. E **61**, R1001 (2000).
[9] H. Haken, *Principles of Brain Functioning* (Springer-Verlag, Berlin, 1996).
[10] G.B. Ermentrout and N. Koppel, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **15**, 215 (1984); L. Strogatz and R.E. Mirollo, J. Phys. A **21**, L699 (1988).
[11] G.V. Osipov and M.M. Sushchik, Phys. Rev. E **58**, 7198 (1998).
[12] S.C. Manrubia and A.S. Mikhailov, Phys. Rev. E **60**, 1579 (1999); R.V. Mendes, Phys. Lett. A **257**, 132 (1999).
[13] V.I. Nekorkin, V.A. Makarov, and M.G. Velarde, Phys. Rev. E **58**, 5742 (1998).
[14] T.E. Vadivasova, G.I. Strelkova, and V.S. Anishchenko, Phys. Rev. E **63**, 036225 (2001).
[15] H. Gang, T. Ditzinger, C.Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).
[16] A.S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[17] A.C. Scott, Rev. Mod. Phys. **47**, 487 (1975).
[18] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[19] The distribution interval of coupling strength is determined as $\Delta = g_{max} - g_{min}$, where g_{min} is fixed at 0.005 but g_{max} and the mean level $(g_{min}+g_{max})/2$ are varied.
[20] H. Treutlein and K. Schulten, Eur. Biophys. J. **13**, 355 (1986).
[21] A.N. Malakhov, *Fluctuations in Autooscillatory Systems* (Nauka, Moscow, 1968).
[22] R.L. Stratonovich, *Topics in the Theory of the Random Noise* (Gordon and Breach Science Publishers, New York, 1981).
[23] We use the instantaneous phase introduced as $\Phi_j(t) = 2\pi(t - t_k/t_{k+1} + t_k) + 2\pi k$, where t_k is the time of kth firing defined in simulations by the threshold crossing of $x_j(t)$ at $x = 1.0$.